Mutual Chern-Simons theory for Z_2 topological order

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We study several different Z_2 topological ordered states in frustrated spin systems. The effective theories for those different Z_2 topological orders all have the same form—a Z_2 gauge theory which can also be written as a mutual $U(1)\times U(1)$ Chern-Simons theory. However, we find that the different Z_2 topological orders are reflected in different projective realizations of lattice symmetry in the same effective mutual Chern-Simons theory. This result is obtained by comparing the ground-state degeneracy, the ground-state quantum numbers, the gapless edge state, and the projective symmetry group of quasiparticles calculated from the slave-particle theory and from the effective mutual Chern-Simons theories. Our study reveals intricate relations between topological order and symmetry.

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I. INTRODUCTION

After the discovery of fractional quantum Hall effectm,[1] we realized that a different kind of orders beyond Landau's symmetry breaking paradigm is possible. This different kind order is called topological order[2,3] for gapped states and quantum order[4] for general states. The orders reflect patterns of long-range entanglements in the ground state.

Gapped Z_2 spin liquids have the simplest kind of topological order—Z_2 topological order.[5,6] Those topological ordered states may appear in frustrated spin systems or dimmer models.[5–11] Physically, the topological orders can be (partially) characterized by robust ground-state degeneracy.[6,12] The low-energy effective theory for those Z_2 topologically ordered states is a Z_2 gauge theory.

Topological order is a property of a many-body ground state that is robust against any perturbations, even those perturbations that break all the symmetries. In this paper, we would like to study the interplay between topological order and symmetry. We would like to find out how to characterize topological ordered states that also have certain symmetries.

Recently, it was found that for spin liquids with all the lattice symmetries (such as lattice translation and rotation symmetry), there can be hundreds different Z_2 topological orders.[4,9] We will call those topological orders symmetric topological orders. It is shown that the different symmetric Z_2 topological orders can be characterized by different project symmetry groups (PSG). So those symmetric topological orders are good examples to study the relation between topological order and symmetry.

Here, using two examples of Z_2 topological orders (we call them $Z2A$ and the $Z2E$ states in below), we would like to study their low-energy effective theories and ask how different symmetric Z_2 topological orders are reflected in low-energy effective theories. It was pointed out that the Z_2 gauge theory can be described by effective mutual $U(1)\times U(1)$ Chern-Simons (CS) theories.[13–15] We find that the two types of Z_2 topological orders ($Z2A$ and $Z2E$ states) can indeed be described by the same effective mutual $U(1)\times U(1)$ CS theories. In the effective mutual CS theories, the lattice symmetry is realized projectively. It turns out that the two different symmetric Z_2 topological orders have different projective realizations of the lattice symmetries.

After knowing how lattice symmetries are realized in the effective mutual CS theory, we can use the effective theory to calculate the numbers of degenerate ground states and their quantum numbers under those lattice symmetries. To confirm those results from effective theory, the projective construction (the slave-particle theory)[16,17] is used to calculate the ground-state degeneracies, the ground-state quantum numbers, and the PSGs of quasiparticles. Those results agree with the results from the effective mutual CS theories. Furthermore, we also used the effective mutual CS theories to study gapless edge states for the two types of Z_2 topologically ordered states.

II. PROJECTIVE CONSTRUCTION OF MANY-SPIN WAVE FUNCTIONS

The key to understand topological orders is to construct states that can have long-range quantum entanglements. The projective construction introduced in the study of high T_c superconductors is a powerful way to construct such states.[16–19] In this section, we will briefly review the projective construction of Z_2 topologically ordered states.

A spin-1/2 model can be viewed as a hard-core-boson model if we identify $|\downarrow\rangle$ state as a zero-boson state $|0\rangle$ and $|\uparrow\rangle$ state as a one-boson state $|1\rangle$. In following parts we will use the boson-picture to describe our model.

We first introduce a "mean-field" fermion Hamiltonian[4]

$$H_{\text{mean}}=\sum_{\langle ij\rangle}(\psi_{I,i}^{\dagger}u_{ij}^{IJ}\psi_{J,j}+\psi_{I,i}^{\dagger}\eta_{ij}^{IJ}\psi_{J,j}^{\dagger}+\text{h.c.}), \tag{1}$$

where $I,J=1,2$. We will use u_{ij} and η_{ij} to denote the 2×2 complex matrices whose elements are u_{ij}^{IJ} and η_{ij}^{IJ}. Let $|\Psi_{\text{mean}}^{(u_{ij},\eta_{ij})}\rangle$ be the ground state of the above free fermion Hamiltonian (i.e., the lowest energy state obtained by filling all the negative-energy levels). Then a many-boson wave function can be obtained through

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$$\Phi_{\text{spin}}^{(u_{ij},\eta_{ij})}(i_1,i_2\ldots) = \langle 0| \prod_{n=1}^{N_{\text{site}}/2} b(i_n)|\Psi_{\text{mean}}^{(u_{ij},\eta_{ij})}\rangle \quad (2)$$

where N_{site} is the number of lattice sites,

$$b(i) = \psi_{1,i}\psi_{2,i}, \quad (3)$$

and i_1, i_2, and … label the location of bosons (up spins). Here, we have assumed that there are $N_{\text{site}}/2$ up spins and $N_{\text{site}}/2$ down spins.

We may view (u_{ij},η_{ij}) as variational parameters and the physical spin-wave function $\Phi_{\text{spin}}^{(u_{ij},\eta_{ij})}(i_1,i_2\ldots)$ as a trial wave function. The trial ground state of a spin Hamiltonian can be obtained by minimizing the average energy $\langle H\rangle$.

First let us consider the following spin Hamiltonian:

$$H_{\text{exact}} = g\sum_i \hat{F}_i, \quad \hat{F}_i = \sigma_i^y\sigma_{i+\hat{x}}^x\sigma_{i+\hat{x}+\hat{y}}^y\sigma_{i+\hat{y}}^x, \quad (4)$$

where $\sigma^{x,y,z}$ are the Pauli matrices and $i=(i_x,i_y)$ labels the site of a square lattice. We find that if we choose the variational parameters to be

$$-\eta_{i,i+\hat{x}} = u_{i,i+\hat{x}} = 1+\tau^z,$$

$$-\eta_{i,i+\hat{y}} = u_{i,i+\hat{y}} = 1-\tau^z, \quad (5)$$

then the spin-wave function Eq. (2) minimizes the average energy. In fact the wave function is the exact ground state of Hamiltonian H_{exact}.[9] It was found that all the excitations above the ground state are gapped and the ground state contains a nontrivial topological order described by a Z_2 effective gauge theory. We will call such a state *Z*2E state.

Ref. 6 introduced another many-spin state on square lattice which is described by

$$u_{i,i+\hat{x}} = u_{i,i+\hat{y}} = -\chi\tau^3,$$

$$u_{i,i+\hat{x}+\hat{y}} = \eta\tau^1 + \lambda\tau^2,$$

$$u_{i,i-\hat{x}+\hat{y}} = \eta\tau^1 - \lambda\tau^2,$$

$$u_{ii} = \upsilon\tau^1, \quad (6)$$

and $\eta_{ij}=0$. However, it is not clear what kind of spin Hamiltonian gives rise to the spin state described by the above variational parameters. Despite this, some physical properties of the spin state were obtained under the assumptions that the state is stable for a certain local spin Hamiltonian.[6] Again, all excitations above the spin state have finite-energy gaps. The spin state is a spin liquid with no spin order. But it contains a nontrivial topological order described by an effective Z_2 gauge theory. So we will call such a spin state *Z*2A state.

Naively, one may expect the *Z*2A and the *Z*2E states to be the same state since both have Z_2 gauge theory as their low-energy effective theory. In the following, we will show that they are different quantum states with different topological orders.

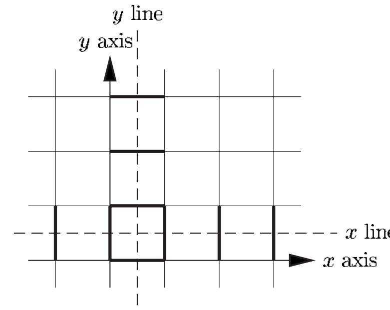

FIG. 1. The links crossing the x and y lines get an additional minus sign.

III. GROUND-STATE DEGENERACY

One way to study a topological order is to study its ground-state degeneracy on a torus. Naively, we expect the *Z*2A and the *Z*2E state to have four degenerate ground states as implied by the effective Z_2 gauge theory. The argument goes as follows.

First, we note that the physical boson wave function $\Phi^{(u_{ij},\eta_{ij})}(\{i_n\})$ is invariant under the following $SU(2)$ gauge transformations[16]

$$(\psi_i,u_{ij},\eta_{ij}) \to (G_i\psi_i, G_iu_{ij}G_j^\dagger, G_i\eta_{ij}G_j^T), \quad (7)$$

where $G_i \in SU(2)$. So the average energy $E(u_{ij},\eta_{ij}) = \langle\Phi^{(u_{ij},\eta_{ij})}|H|\Phi^{(u_{ij},\eta_{ij})}\rangle$ satisfies

$$E(u_{ij},\eta_{ij}) = E(G_iu_{ij}G_j^\dagger, G_i\eta_{ij}G_j^T).$$

Next we assume that $(\bar{u}_{ij},\bar{\eta}_{ij})$ give rise to a (variational) ground state of a Hamiltonian. We would like to show that the following four *Ansätze*

$$u_{ij}^{(m,n)} = (-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{u}_{ij},$$

$$\eta_{ij}^{(m,n)} = (-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{\eta}_{ij} \quad (8)$$

produce four degenerate ground states. Here $m,n=0,1$. $s_x(ij)$ and $s_y(ij)$ have values 0 or 1. $s_x(ij)=1$ if the link ij crosses the x line (see Fig. 1) and $s_x(ij)=0$ otherwise. Similarly, $s_y(ij)=1$ if the link ij crosses the y line and $s_y(ij)=0$ otherwise. Physically, the degenerate states arise from adding π flux through the two holes of the torus. The values of $m,n=0,1$ reflect the presence or the absence of the π flux in the two holes.

We note that $(u_{ij}^{(0,0)},\eta_{ij}^{(0,0)})$ represents the ground state. We also note that $(u_{ij}^{(m,n)},\eta_{ij}^{(m,n)})$ with different m and n are *locally* gauge equivalent. This is because, on an infinite system, the change, say, $u_{ij}\to(-)^{ms_x(ij)}(-)^{ms_y(ij)}u_{ij}$ can be generated by an $SU(2)$ gauge transformation $u_{ij}\to W_iu_{ij}W_j^\dagger$, where $W_i=(-)^{m\Theta(i_x)}(-)^{n\Theta(i_y)}$, and $\Theta(n)=1$ if $n>0$ and $\Theta(n)=0$ if $n\le 0$. As a result, $E(\bar{u}_{ij},\bar{\eta}_{ij})=E(u_{ij}^{(m,n)},\eta_{ij}^{(m,n)})$. On the other hand, on a torus, $(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})$ with different m and n are not gauge equivalent in the global sense. There is no $SU(2)$ gauge transformation defined on the torus that connects those *Ansätze*. So the four *Ansätze* give rise to four different degenerate states. This is how we obtain the fourfold ground-state degeneracy for the Z_2 states.

However, the above argument is valid only for even by even lattice. For odd by odd lattice, the argument breaks down. To understand the failure of the above argument, let us construct the mean-field ground state more carefully.

Let us start with a simple case of the Z2A state. For the *Ansatz* Eq. (6), the mean-field Hamiltonian in momentum space becomes

$$H_{\text{mean}}(\mathbf{k}) = \sum_{\mathbf{k}} (\psi_{1\mathbf{k}}^\dagger, \psi_{2\mathbf{k}}^\dagger) M \begin{pmatrix} \psi_{1\mathbf{k}} \\ \psi_{2\mathbf{k}} \end{pmatrix} = \sum_k \varepsilon(\mathbf{k}) \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} - \sum_k \varepsilon(\mathbf{k}) \beta_{\mathbf{k}}^\dagger \beta_{\mathbf{k}}, \tag{9}$$

where

$$M = 2\chi(\cos k_x + \cos k_y)\tau^3 + [2\eta \cos(k_x + k_y) + 2\eta \cos(k_x - k_y) + v]\tau^1 + [2\lambda \cos(k_x + k_y) - 2\lambda \cos(k_x - k_y)]\tau^2,$$

and

$$\varepsilon(\vec{k}) = \sqrt{4\chi^2(\cos k_x + \cos k_y)^2 + [2\eta \cos(k_x + k_y) + 2\eta \cos(k_x - k_y) + v]^2 + [2\lambda \cos(k_x + k_y) - 2\lambda \cos(k_x - k_y)]^2}.$$

Here $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are diagonalized quasiparticles operators

$$\alpha_{\mathbf{k}} = (a\psi_{1\mathbf{k}} + \psi_{2\mathbf{k}})/\sqrt{1+a^2},$$

$$\beta_{\mathbf{k}} = (b\psi_{1\mathbf{k}} + \psi_{2\mathbf{k}})/\sqrt{1+b^2},$$

where a and b are the functions of k_x and k_y. The mean-field ground state is obtained by filling all the negative levels and is given by

$$|\Psi_{\text{mean}}\rangle = \prod_k \beta_{\mathbf{k}}^\dagger |0\rangle_\psi,$$

where the state $|0\rangle_\psi$ is defined through $\psi_{\mathbf{k}}|0\rangle_\psi = 0$. (Note that all the particles $\alpha_{\mathbf{k}}$ has positive energy and all the particles $\beta_{\mathbf{k}}$ has negative energy.) Since $\beta_{\mathbf{k}}^\dagger$ is linear combination of $\psi_1^\dagger$ and $\psi_2^\dagger$ and there are $L_x \times L_y$ different $\mathbf{k}$ levels, the mean-field state $|\Psi_{\text{mean}}\rangle$ contains $L_x \times L_y$ number of fermions. Here $L_{x,y}$ are sizes of the lattice in the x and y directions.

Clearly, when both L_x and L_y are odd, $|\Psi_{\text{mean}}\rangle$ contains an odd number of fermions. Such a mean-field state does not correspond to any physical spin state since the corresponding spin-wave function Eq. (2) vanishes. [Note that Eq. (2) is a projection to the subspace with 0 or 2 fermions per site.] To get a nonzero physical spin-wave function we need to start with a mean-field state with one extra fermion in the empty α band (or a hole in the filled β band). But by choosing different states for the extra fermion (or the hole), we can obtain many different spin-wave functions which are nearly degenerate. So when both L_x and L_y are odd, the excitations in the Z2A state are gapless, or we may say that the Z2A state has infinite degeneracy. Physically, the Z2A state on odd by odd lattice always contains an unpaired spinon. The different states of the unpaired spinon give rise to the infinite degeneracy.

When one of $L_{x,y}$ is even, the mean-field state $|\Psi_{\text{mean}}\rangle$ gives rise to a nonzero physical spin state. There is no unpaired spinon, and the excitations are gaped. Each *Ansatz* $u_{ij}^{(m,n)}$ produces a single physical spin state, and the Z2A state has fourfold degeneracy on a torus with an even number of lattice sites.

Because the spin Hamiltonian is translation invariant, the ground states carry definite crystal momentum. To calculate the crystal momentum, we note that in the $(m,n)=(0,0)$ sector described by the *Ansatz* $u_{ij}^{(0,0)}$, the fermion wave function satisfies the periodic boundary condition. So (k_x,k_y) are quantized as $(k_x,k_y)=(n_x\frac{2\pi}{L_x}, n_y\frac{2\pi}{L_y})$ where $n_{x,y}$ are integers. Moreover, the spin state produced by the *Ansatz* $u_{ij}^{(0,0)}$ has the following crystal momentum:

$$K_x = \sum k_x = \sum_{n_x=1}^{L_x} \sum_{n_y=1}^{L_y} n_x \frac{2\pi}{L_x} = \frac{L_y L_x (L_x+1)}{2} \frac{2\pi}{L_x},$$

$$K_y = \sum k_y = \sum_{n_x=1}^{L_x} \sum_{n_y=1}^{L_y} n_y \frac{2\pi}{L_y} = \frac{L_x L_y (L_y+1)}{2} \frac{2\pi}{L_y}.$$

We would like to point out that the above crystal momentum is actually the crystal momentum of the mean-field state. However, the even-fermion-per-site projection commutes with the translation operator, and thus the crystal momentum is unchanged by projection.

When m and/or n are equal to 1, the fermion wave function is antiperiodic in the y and/or x directions. In the case, k_y and/or k_x are quantized as $(n_y+\frac{1}{2})\frac{2\pi}{L_y}$ and/or $(n_x+\frac{1}{2})\frac{2\pi}{L_x}$. The crystal momentum of the spin state produce by the *Ansatz* $u_{ij}^{(m,n)}$ can be calculated in the similar fashion. For example in the $(m,n)=(1,1)$ sector, the crystal momentum is given by

$$K_x = \sum_{n_x=1}^{L_x} \sum_{n_y=1}^{L_y} \left(n_x + \frac{1}{2}\right) \frac{2\pi}{L_x} = \frac{L_y L_x (L_x+2)}{2} \frac{2\pi}{L_x},$$

$$K_y = \sum_{n_x=1}^{L_x} \sum_{n_y=1}^{L_y} \left(n_y + \frac{1}{2}\right) \frac{2\pi}{L_y} = \frac{L_x L_y (L_y+2)}{2} \frac{2\pi}{L_y}.$$

The results are summarized in the Table I.

IV. TOPOLOGICAL PROPERTIES FOR THE EXACT SOLUBLE MODEL

To understand the topological order in the $Z2E$ state of the exact soluble model, we would like to calculate the ground-state degeneracy and ground-state crystal momenta of the $Z2E$ state. Just like the $Z2A$ state discussed in the last section, one can construct many-spin wave functions of the degenerate ground states from the mean-field *Ansätze* Eq. (8) with (u_{ij},η_{ij}) given by Eq. (5). The four mean-field *Ansätze* $(u_{ij}^{(m,n)},\eta_{ij}^{(m,n)})$ can potentially give rise to four degenerate ground states. But some time, the mean-field ground state contains odd numbers of fermions. In this case, the corresponding mean-field *Ansatz* does not lead to physical spin-wave function.

TABLE I. Crystal momenta (K_x,K_y) of the four ground states, $(m,n)=(0,0),(0,1),(1,0),(1,1)$, of the $Z2A$ spin liquid on three different lattices, $(L_x,L_y)=(\text{even},\text{even})$,(even,odd), and (odd,even).

(K_x,K_y)	(ee)	(eo)	(oe)	(oo)
(00)	(0,0)	$(\pi,0)$	$(0,\pi)$	—
(01)	(0,0)	$(\pi,0)$	(0,0)	—
(10)	(0,0)	(0,0)	$(0,\pi)$	—
(11)	(0,0)	(0,0)	(0,0)	—

To calculate the fermion number in the mean-field ground state, one can write down the mean-field fermion Hamiltonian in momentum space

$$\begin{aligned}H_{\text{mean}}(\mathbf{k}) &= \sum_{k>0}(\psi_{1\mathbf{k}}^\dagger,\psi_{1,-\mathbf{k}})\begin{pmatrix}\cos k_x & \mathrm{i}\sin k_x\\ -\mathrm{i}\sin k_x & -\cos k_x\end{pmatrix}\begin{pmatrix}\psi_{1\mathbf{k}}\\ \psi_{1,-\mathbf{k}}^\dagger\end{pmatrix} + \sum_{k>0}(\psi_{2\mathbf{k}}^\dagger,\psi_{2,-\mathbf{k}})\begin{pmatrix}\cos k_y & \mathrm{i}\sin k_y\\ -\mathrm{i}\sin k_y & -\cos k_y\end{pmatrix}\begin{pmatrix}\psi_{2\mathbf{k}}\\ \psi_{2,-\mathbf{k}}^\dagger\end{pmatrix} + \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=0,k_y=0}\\ &\quad - \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=\pi,k_y=\pi} + \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=0,k_y=0} - \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=\pi,k_y=\pi} + \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=0,k_y=\pi} - \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=\pi,k_y=0} - \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=0,k_y=\pi}\\ &\quad + \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=\pi,k_y=0} = \sum_{k>0}[\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + \alpha_{-\mathbf{k}}^\dagger\alpha_{-\mathbf{k}}] + \sum_{k>0}[\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}} + \beta_{-\mathbf{k}}^\dagger\beta_{-\mathbf{k}}] + \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=0,k_y=0} - \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=\pi,k_y=\pi}\\ &\quad + \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=0,k_y=0} - \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=\pi,k_y=\pi} + \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=0,k_y=\pi} - \psi_{1\mathbf{k}}^\dagger\psi_{1\mathbf{k}}|_{k_x=\pi,k_y=0} - \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=0,k_y=\pi} + \psi_{2\mathbf{k}}^\dagger\psi_{2\mathbf{k}}|_{k_x=\pi,k_y=0},\end{aligned} \tag{10}$$

with

$$\begin{pmatrix}\alpha_{\mathbf{k}}\\ \alpha_{-\mathbf{k}}^\dagger\end{pmatrix} = \exp\left[-\mathrm{i}k_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}\psi_{1\mathbf{k}}\\ \psi_{1,-\mathbf{k}}^\dagger\end{pmatrix},$$

$$\begin{pmatrix}\beta_{\mathbf{k}}\\ \beta_{-\mathbf{k}}^\dagger\end{pmatrix} = \exp\left[-\mathrm{i}k_y\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}\psi_{2\mathbf{k}}\\ \psi_{2,-\mathbf{k}}^\dagger\end{pmatrix}.$$

Here $\mathbf{k}=0$ means that $(k_x,k_y)=(0,0)$, $(0,\pi)$, $(\pi,0)$, or (π,π), and $\mathbf{k}>0$ means that $k_y>0$ or $k_y=0$, $k_x>0$, *and* $\mathbf{k}\neq 0$.

We note that both α band and β band have a positive energy $E_{\mathbf{k}}=1$. $\alpha_{\pm\mathbf{k}}$, $\beta_{\pm\mathbf{k}}$ will annihilate the mean-field ground state $|\Psi_{\text{mean}}\rangle$,

$$\alpha_{\pm\mathbf{k}}|\Psi_{\text{mean}}\rangle = 0, \quad \beta_{\pm\mathbf{k}}|\Psi_{\text{mean}}\rangle = 0.$$

It needs to be pointed out that the above formulas for the mean-field fermion Hamiltonian are valid only for even-by-even lattice with periodic boundary condition, i.e., $(m,n)=(0,0)$. For other cases (even-by-even lattice with antiperiodic boundary conditions, and even-by-odd, odd-by-even, and odd-by-odd lattices with both periodic boundary condition and antiperiodic boundary conditions), one or more of the four high-symmetry points at momentum space $\mathbf{k}^*=(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π) are absent, which is shown in the table in the Appendix.

We also note that, for $\mathbf{k}\neq 0$,

$$\alpha_{\boldsymbol{k}} = u_{\boldsymbol{k}}\psi_{1,\boldsymbol{k}} + v_{\boldsymbol{k}}\psi_{1,-\boldsymbol{k}}^\dagger$$

$$\alpha_{-\boldsymbol{k}}^\dagger = -v_{\boldsymbol{k}}^*\psi_{1,\boldsymbol{k}} + u_{\boldsymbol{k}}^*\psi_{1,-\boldsymbol{k}}^\dagger.$$

The condition $\alpha_{\boldsymbol{k}}|\Phi_{\text{mean}}\rangle=\alpha_{-\boldsymbol{k}}|\Phi_{\text{mean}}\rangle=0$ implies that (if we only consider the $\boldsymbol{k}$ and $-\boldsymbol{k}$ levels)

$$|\Phi_{\text{mean}}\rangle = (v_{\boldsymbol{k}} + u_{\boldsymbol{k}}\psi_{1,-\boldsymbol{k}}^\dagger\psi_{1,\boldsymbol{k}}^\dagger)|0\rangle.$$

We see that $\mathbf{k}\neq 0$ levels always contribute even numbers of fermions. Also, since $v_{\boldsymbol{k}}+u_{\boldsymbol{k}}\psi_{1,-\boldsymbol{k}}^\dagger\psi_{1,\boldsymbol{k}}^\dagger$ carries 0 momentum, we see that the contribution to the total momentum from the $\mathbf{k}\neq 0$ levels is zero.

Thus to determine if the mean-field ground state contains even or odd number of ψ fermions, we only need to examine the occupation on the four $\boldsymbol{k}=0$ momentum points: $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π). The Hamiltonian on those four points is contained in Eq. (10). All the negative-energy levels are filled in the mean-field ground state. On an even by even lattice and for the $(m,n)=(0,0)$ *Ansatz*, all the momenta $(\pi,0)$, $(0,\pi)$, and (π,π) are allowed. Thus the $(\pi,0)$ level and the (π,π) level each is occupied by a ψ_1 fermion, and the $(0,\pi)$ level and the (π,π) level each is occupied by a ψ_2 fermion. The total momentum of the ground state is (π,π). Such a mean-field ground state has even numbers of fermions. It will survive the projection and lead to a physical spin ground state. Other situations can be calculated in the same way. Here we only summarize the result: on an even by even lattice, there exist four different degenerate ground states.

TABLE II. Crystal momenta of the degenerate ground states, $(m,n)=(0,0),(0,1),(1,0),(1,1)$, of the Z2E spin liquid on four different lattices, $(L_x,L_y)=(\text{even},\text{even})$, (even,odd), (odd,even), and (odd,odd).

(K_x,K_y)	(ee)	(eo)	(oe)	(oo)
(00)	(π,π)	—	—	—
(01)	(0,0)	—	(0,0)	(0,0)
(10)	(0,0)	(0,0)	—	(0,0)
(11)	(0,0)	(0,0)	(0,0)	—

However, on other kinds of lattice (even by odd, odd by even, and odd by odd), there exist only two different ground states. The other two states are projected out since the mean-field ground states contain odd numbers of fermions. The crystal momenta of the degenerate ground states can also be calculated which are summarized in Table II.

V. MUTUAL $U(1)\times U(1)$ CS THEORY

In Secs. III and IV, we have calculated the topological properties of the Z2A and the Z2E states. Due to their different topological properties, we find that the two states have different topological orders. Then an important issue is to find the low-energy effective theories that describe the two different topological orders. We find that a mutual $U(1)\times U(1)$ CS theory with different projective realizations of the lattice symmetry can describe the two kind of topological orders. We reach the conclusion by comparing the topological properties of the mutual $U(1)\times U(1)$ CS theory with those of the Z2A and the Z2E states. All the topological properties, including topological degeneracy, quantum numbers, and edge states agree, indicating the equivalence between the Z_2 topological states on lattice and the mutual $U(1)\times U(1)$ CS theory.

A. Mutual $U(1)\times U(1)$ CS theory

First we introduce the Lagrangian for the mutual $U(1)\times U(1)$ CS theory

$$\mathcal{L}_{\text{eff}}=-\frac{1}{4e_a^2}(f_{\mu\nu})^2-\frac{1}{4e_A^2}(F_{\mu\nu})^2 \tag{11}$$

$$+\frac{1}{\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda+\mathrm{i}a^\mu j_\mu+\mathrm{i}A^\mu J_\mu, \tag{12}$$

where $f_{\mu\nu}$ is the gauge-field strength for gauge field a_λ and $F_{\mu\nu}$ is the gauge-field strength for gauge field A_μ. The excitations are described by the currents, which are defined as $j_\mu=(j_i,\rho_a)$ and $J_\mu=(J_i,\rho_A)$. The gauge charges of a_μ and A_μ are quantized as integers. The mutual CS theory in above equations has been used to study the topological order in frustrated Josephson-junction arrays.[14,15] In addition, similar mutual CS theory was proposed to be the effective gauge theory of doped Mott insulator[20,21] for high T_c superconductors.

From the equation motions for a_λ and A_λ,

$$-\frac{1}{2e_a}(\partial_\mu f_{\mu\lambda})+\frac{1}{\pi}\epsilon^{\mu\nu\lambda}F_{\mu\nu}=-\mathrm{i}j_\mu,$$

$$-\frac{1}{2e_A^2}(\partial_\mu F_{\mu\lambda})+\frac{1}{\pi}\epsilon^{\mu\nu\lambda}f_{\mu\nu}=-\mathrm{i}J_\mu,$$

we find that a $U(1)$ charge for gauge field A_μ induces flux of gauge field a_μ. As a result, the $U(1)$ charge for gauge field A_μ and the $U(1)$ charge for gauge field a_μ have a semionic mutual statistics. That is, moving an A_μ charge around an a_μ charge generates a phase π. This catches the key topological property for the Z_2 spin liquid. It is well known that the Z_2 spin liquid states contain Z_2 vortex and Z_2 charge excitations. Moreover, the Z_2 vortex and the Z_2 charge have semionic mutual statistics between them. So we will propose that the mutual Chern-Simons theory in Eq. (11) describes a Z_2 gauge theory. The A_μ charge can be identified as the Z_2 charge and the a_μ charge as the Z_2 vortex.

Furthermore, the energy gap for both of the gauge fields comes from the mutual CS term

$$m_a\sim e_ae_A,\quad m_A\sim e_ae_A.$$

The mutual $U(1)\times U(1)$ CS theory describes a gapped topological state. This also agrees with the Z_2 topological states where all excitations are gapped.

However, we have two kinds of Z_2 topological orders Z2A and Z2E. How can the two different Z_2 topological orders be described by the same $U(1)\times U(1)$ CS theory? In the following we will show that two different Z_2 topological orders are described by the same $U(1)\times U(1)$ CS theory but with different realizations of the lattice symmetry.

To obtain two different realizations of lattice symmetry, we note that Z_2 vortices for the exactly soluble model (the Z2E state) live on the even plaquettes. The vortices on the odd plaquettes are actually the Z_2 charge.[9,16] So under a translation by one lattice spacing, a Z_2 vortex is changed into a Z_2 charge! So in the mutual $U(1)\times U(1)$ CS theory that describes the Z2E state, a_μ and A_i must exchange under the translation by one lattice spacing.

Also, the Z2A state contains π flux through each square. This π flux also affects how a_μ is transformed under translation. To see this, let us consider two Wilson loop operators $W_1=e^{\mathrm{i}\oint_{C_1}dy a_y}$ and $W_2=e^{\mathrm{i}\oint_{C_2}dy a_y}$ along two loops C_1 and C_2. Both loops wrap around the torus in the y direction. However, the loop C_2 is displaced from the loop C_1 by one lattice constant in the x direction. In the following, we will assume the lattice constant is $a=1$. Due to the π flux through each square, we see that $W_2=(-)^{L_y}W_1$, where L_y is the length of the torus in the y direction. So under a translation by one lattice constant in the x direction, a_y must change to $a_y+\pi$, to account for the change in the Wilson loop.

The above discussion motivates us to define two types of mutual $U(1)\times U(1)$ CS theories which have different realizations of translation symmetries. Let T_x and T_y be the translations by one lattice spacing in the x and y directions, respectively. The first type of the mutual $U(1)\times U(1)$ CS theory is denoted as $Z2$A type which describes the $Z2$A state. The π flux makes the gauge fields transform nontrivially under translations

$$T_x^{-1}A_xT_x = A_x, \quad T_y^{-1}A_xT_y = A_x + \pi,$$
$$T_x^{-1}A_yT_x = A_y + \pi, \quad T_y^{-1}A_yT_y = A_y,$$
$$T_x^{-1}a_xT_x = a_x, \quad T_y^{-1}a_xT_y = a_x + \pi,$$
$$T_x^{-1}a_yT_x = a_y + \pi, \quad T_y^{-1}a_yT_y = a_y. \tag{13}$$

Since the translation T_x (T_y) may shift A_y (a_x) by π, this reproduces the different patterns of crystal momenta of the degenerate ground states on different lattices.

The other type of the mutual CS theory is denoted as $Z2$E type that describes the $Z2$E state. It has no flux. However, the gauge fields still transform nontrivially under translations

$$T_i^{-1}A_jT_i = a_j, \quad T_i^{-1}a_jT_i = A_j, \quad i = x,y. \tag{14}$$

A_i and a_i will exchange under a translation operation by one lattice spacing.

B. Topological degeneracy

In Secs. V B and V C, we will calculate the topological properties of the above two types of mutual CS theory. First, we calculate the topological degeneracy for the ground states. In the temporal gauge, $A_0=0$, and on an even-by-even lattice, the fluctuations A_i and a_i are periodic. We can expand them as

$$(A_x,A_y) = \left(\frac{1}{L_x}\Theta_x + \sum_{\mathbf{k}} A_{\mathbf{k}}^x e^{i\check{x}\cdot\mathbf{k}}, \frac{1}{L_y}\Theta_y + \sum_{\mathbf{k}} A_{\mathbf{k}}^y e^{i\check{x}\cdot\mathbf{k}}\right), \tag{15}$$

$$(a_x,a_y) = \left(\frac{1}{L_x}\theta_x + \sum_{\mathbf{k}} a_{\mathbf{k}}^x e^{i\check{x}\cdot\mathbf{k}}, \frac{1}{L_y}\theta_y + \sum_{\mathbf{k}} a_{\mathbf{k}}^y e^{i\check{x}\cdot\mathbf{k}}\right), \tag{16}$$

where $\mathbf{k}=(k_x,k_y)=(\frac{2\pi}{L_x}n_x, \frac{2\pi}{L_y}n_y)$ where $n_{x,y}$ are integers. $(A_{\mathbf{k}}^x,A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x,a_{\mathbf{k}}^y)$ are the gauge fields with nonzero momentum, and (Θ_x,Θ_y) and (θ_x,θ_y) are the zero modes with zero momentum for the gauge fields A_i and a_i. Because the existence of the mass gap, the degree freedoms for gauge fields with nonzero momentum $(A_{\mathbf{k}}^x,A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x,a_{\mathbf{k}}^y)$ have nothing to do with the low-energy physics. It is the degree freedoms of zero momentum (Θ_x,Θ_y) and (θ_x,θ_y) that determine the low-energy physics. The effective Lagrangian Eq. (11) determines the dynamics of (Θ_x,Θ_y) and (θ_x,θ_y) which corresponds to two particles on a plane with a finite magnetic field. (Θ_x,θ_y) are the coordinates of the first particle, and (Θ_y,θ_x) are the coordinates of the second particle. Thus we map the original mutual $U(1)\times U(1)$ CS theory to a quantum mechanics model of two particles (see Appendix). The energy spectrum for the quantum mechanics model can be solved easily. The lowest energy levels for the above model reveal the topological characters for the ground states. The degeneracy for (Θ_x,θ_y) degrees of freedom and the degeneracy for (Θ_y,θ_x) degrees of freedom are given as $D_{(\Theta_x,\theta_y)}=2$ and $D_{(\Theta_y,\theta_x)}=2$. For both the $Z2$A-type and the $Z2$E-type CS theories, there exist four degenerate ground states

$$D = D_{(\Theta_x,\theta_y)}D_{(\Theta_y,\theta_x)} = 2 \times 2 = 4. \tag{17}$$

However, the above result only applies to even-by-even lattice. For other cases (even by odd, odd by even, and odd by odd), the situations are changed. We will discuss those more complicated cases in the Appendix. We find that for the $Z2$A-type mutual CS theory, the ground-state degeneracy remain to be four for even-by-odd and odd-by-even lattices. For the $Z2$E-type mutual CS theory, the ground-state degeneracy becomes two for even-by-odd, odd-by-even, and odd-by-odd lattices.

One way to understand the later result is to note that if L_x is odd then one gauge field will turn into the other one as we go around the lattice along the x direction. Thus the gauge fields have a twisted boundary condition

$$A_i(x+L_x,y) = a_i(x,y), \quad a_i(x+L_x,y) = A_i(x,y).$$

This twisted boundary condition means that A_μ and a_μ can be viewed as a single gauge field on a lattice whose size is doubled in the x direction. There are only two zero modes in the mode expansion. As a result the ground-state degeneracy on even-by-odd, odd-by-even, and odd-by-odd is reduced to two. We can also use the CS theories to calculate the crystal momenta of the ground states (see Appendix). The results agree with those in Tables I and II.

It is well known that for a $U(1)\times U(1)$ CS theory in continuum limit, the ground-state degeneracy is determined by the Chern-Simons coefficients and the genus of the manifold. But this result is obtained with an assumption that the $U(1)$ gauge fields satisfy a simple periodic boundary condition. However, for certain realizations of lattice translation symmetries, we see that the $U(1)$ gauge fields satisfy certain nontrivial periodic boundary conditions, depending on if the lattice is even by even or odd by odd etc. This leads to different ground-state degeneracies even though the Chern-Simons coefficients are not changed.

C. Edge states

We can also use the mutual $U(1)\times U(1)$ CS theories to study edge excitations. First, let us consider the exact soluble model (4) on a finite $L_x\times L_y$ lattice with a periodic boundary condition only along the y direction. The lattice has two edges along the y direction located at $i_x=0$ and $i_x=L_x$. Such a lattice model can be obtained from the periodic lattice model (4) by setting $g=0$ for a column of plaquettes. The resulting model is still exactly soluble. We find that the ground states have $\sim 2^{L_y}$-fold degeneracy which arises from $\sigma_i^y\sigma_{i+\check{x}}^x\sigma_{i+\check{x}+\check{y}}^y\sigma_{i+\check{y}}^x=\pm 1$ on the column of plaquettes with $g=0$. Those degenerate states can be viewed as gapless edge excitations on the two boundaries. Since there are $2L_y$ edge sites, we find that there are $\sqrt{2}$ edge states per edge site, indicating that the gapless edge states are described by Majorana fermions. Indeed, the gapless edge excitations can be mapped to a Majorana fermion system exactly.

To obtain the gapless edge states from the mutual CS theories, we introduce

$$a_{+,\mu}=A_\mu+a_\mu, \quad a_{-,\mu}=A_\mu-a_\mu$$

and rewrite the mutual $U(1)\times U(1)$ CS effective theory as

$$\mathcal{L}_{\text{eff}}=\frac{1}{4\pi}a_{+,\mu}\partial_\nu a_{+,\lambda}\epsilon^{\mu\nu\lambda}-\frac{1}{4\pi}a_{-,\mu}\partial_\nu a_{-,\lambda}\epsilon^{\mu\nu\lambda}+\ \ldots . \tag{18}$$

The charges of A_μ and a_μ are quantized as integers. Converting the A_μ and a_μ charges to the $a_{+,\mu}$ and $a_{-,\mu}$ charges, we find that the $a_{+,\mu}$ and $a_{-,\mu}$ charges are still quantized as integers. However, (1/2,1/2) charge for the $a_{+,\mu}$ and $a_{-,\mu}$ field is also allowed.

The mutual CS theory [Eq. (18)] has one right-moving and one left-moving branch of edge excitations. The two branches of the edge excitations are described by the following one-dimensional (1D) fermion theory[16]

$$\mathcal{L}_{\text{edge}}=\psi_R^\dagger(\partial_t-v\partial_x)\psi_R+\psi_L^\dagger(\partial_t+v\partial_x)\psi_L+\ldots$$

at low energies, where (…) represent terms that are consistent with the underlying symmetries of the lattice model. ψ_R carries a unit of a_+ charge and ψ_- a unit of a_- charge. We note that the A_μ and a_μ charges, as the Z_2 charge and the Z_2 vortex, are conserved only mod 2. So (…) may contain terms that change (a_+,a_-) charge by (1,1) and (1,−1). Thus, the following terms

$$a\psi_R\psi_L+b\psi_R\psi_L^\dagger+h.c.$$

are allowed in the low-energy effective Lagrangian. The additional terms will open an energy gap for the edge excitations and one may conclude that the Z2E topological ordered state has no gapless edge excitations in general.

However, the above conclusion is not quite correct. We see that although the presence of the edge breaks the translation symmetry in the x direction, the finite system still has the translation symmetry in the y direction. Under the translation in the y direction by lattice spacing, A_μ and a_μ are exchanged, or $(a_{+,\mu},a_{-,\mu})$ are changed into $(a_{+,\mu},-a_{-,\mu})$. So the translation in the y direction changes the sign of the a_- charge and hence changes ψ_L to $\psi_L^\dagger$. As a result, only the following term

$$a\psi_R(\psi_L+\psi_L^\dagger)+h.c.$$

can be added to the edge effective Lagrangian, which does not break the translation symmetry along the edge.

Introducing Majorana fermions

$$\psi_R=\lambda_R+\mathrm{i}\,\eta_R, \quad \psi_L=\lambda_L+\mathrm{i}\,\eta_L,$$

we can rewrite the edge effective Lagrangian as

$$\mathcal{L}_{\text{edge}}=\lambda_R(\partial_t-v\partial_x)\lambda_R+\eta_R(\partial_t-v\partial_x)\eta_R+\lambda_L(\partial_t+v\partial_x)\lambda_L+\eta_L(\partial_t+v\partial_x)\eta_L+2(a\lambda_R\lambda_L+\mathrm{i}a\lambda_R\eta_L+h.c.).$$

The $a\lambda_R\lambda_L+\mathrm{i}a\lambda_R\eta_L$ term gaps a pair of Majorana fermions and leave the other pair gapless. So the Z2E state has right-moving and left-moving gapless edge excitations described by Majorana fermions, provided that the edge is in the x or y direction. The presence of the translation symmetry in the x or y direction is crucial for the existence of the gapless edge excitations for the Z2E-type mutual $U(1)\times U(1)$ CS theory and the exact soluble model.

For the Z2A state, although the low-energy effective theory has the same form as the exactly soluble model, the translation does not induce the exchange between A_μ and a_μ. As a result, in general, there are no gapless edge excitations for the Z2A-type mutual $U(1)\times U(1)$ CS theory and the Z2A state.

VI. CONCLUSION

In this paper, two kinds of Z_2 topological ordered states for frustrated spin systems, Z2A and Z2E states, are studied. Using the $SU(2)$ slave-particle theory, we calculate their ground-state degeneracy, their ground-state quantum numbers, their gapless edge state, and the projective symmetry group of their quasiparticles. We propose a mutual $U(1)\times U(1)$ Chern-Simons theory with two different realizations of lattice symmetry as the effective-field theories that describe the two types of topological orders. We show that the effective theories produce the same low-energy physics, including the degeneracy of the ground state and the quantum number for the ground state and the edge states. It turns out that the different Z_2 topological orders are reflected in different realizations of the lattice symmetry in the same effective mutual Chern-Simons theory.

We would like to mention that the Z2A phase appears to be an example of "weak symmetry breaking in dimension 2," while the Z2E phase appears to be an example of "weak symmetry breaking in dimension 1" discussed in Ref. 22. So these two phases are examples of the two basic ways that lattice symmetries and topological structure can be entangled.

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APPENDIX

1. Topological degeneracy for the Z2E state

We have used *Ansätze* $[u_{ij}^{(m,n)},\eta_{ij}^{(m,n)}]=[(-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{u}_{ij},(-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{\eta}_{ij}]$ to describe the four degenerate ground states for the Z2E state. Here $m,n=0,1$. $s_{x,y}(ij)$ have values 0 or 1, with $s_{x,y}(ij)=1$ if the link ij crosses the x or y line (see Fig. 1) and $s_{x,y}(ij)=0$ otherwise.

It is pointed out that the above result of four degenerate ground states is right only for the Z2E state on an even-by-even lattice. On other kinds of lattice (even by odd, odd by even, and odd by odd), there exist only two different ground states. The other two states are projected out since the mean-field ground states contain odd numbers of fermions.

Let us calculate the topological degeneracy for the Z2E state on different lattices in detail. It was pointed out that the total number of the ψ fermions on $\mathbf{k}$ and $-\mathbf{k}$ is always even if

$\mathbf{k}\neq 0$. To determine if the mean-field ground state contains an even or odd number of ψ fermions, we will only pay attention to the occupation on the following four momentum points: $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π).

First, we discuss the topological degeneracy for $Z2$E state on an even by even lattice. For the ground state described by $(m,n)=(0,0)$, the energy levels for both ψ_1 and ψ_2 have positive energies at $\boldsymbol{k}=(0,0)$ [see Eq. (10)]. Thus the $\mathbf{k}=(0,0)$ level is not occupied. We also see from Eq. (10) that, at $\boldsymbol{k}=(0,\pi)$, ψ_1 has a positive energy and ψ_2 has a negative energy. Thus the $\mathbf{k}=(0,\pi)$ level is occupied by a ψ_2 particle. Similarly, we find that the $\mathbf{k}=(\pi,0)$ level is occupied by a ψ_1 particle, the $\mathbf{k}=(\pi,\pi)$ level is occupied by a ψ_1 particle and a ψ_2 particle. Therefore, four particles occupy the points (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π). Because the mean-field ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(0,0)},\eta_{ij}^{(0,0)})}\rangle$ has an even number particles, it survives the even-particle-per-site projection.

Also, the total contribution to the crystal momentum from the $\boldsymbol{k}\neq 0$ levels is zero. Thus the total crystal momentum is determined by the particles that occupy the (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π) levels. We find that the total crystal momentum of the above state is $0\times(0,0)+1\times(0,\pi)+1\times(\pi,0)+2\times(\pi,\pi)=(\pi,\pi)$.

For the ground states described by $(m,n)=(1,0)$, $(m,n)=(0,1)$, and $(m,n)=(1,1)$, none of the high-symmetry points (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π) exist. Thus the ground states have even number particles, so they are all permitted under the even-particle-per-site projection. The total crystal momenta of the above states are all zero.

Therefore, there are four degenerate ground states on even-by-even lattice. One carries crystal momentum (π,π) and the other three carry crystal momentum (0,0). This corresponds to the first column of Table II.

Second, we discuss the topological degeneracy for $Z2$E state on an even-by-odd lattice. For the ground state described by $(m,n)=(0,0)$, the $\boldsymbol{k}=(0,0)$ level is not occupied; the $\boldsymbol{k}=(\pi,0)$ level is occupied by one ψ_1 particle as before. The points $(0,\pi)$ and (π,π) do not exist. As a result, only one particle occupies the high-symmetry points. Because the ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(0,0)},\eta_{ij}^{(0,0)})}\rangle$ has odd number particles, it is forbidden by the even-particle-per-site projection.

For the ground state described by $(m,n)=(0,1)$, the points (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π) do not exist. Thus the ground state has even number particles, so it is permitted by the projection. Such a state carries a (0,0) crystal momentum.

For the ground state described by $(m,n)=(1,0)$, the $\boldsymbol{k}=(0,\pi)$ level is occupied by a ψ_2 particle, and the $\boldsymbol{k}=(\pi,\pi)$ level is occupied by a ψ_1 and a ψ_2 particles. The $(\pi,0)$ and (0,0) points do not exist. As a result, three particles occupy the high-symmetry points. The state is forbidden by the projection.

For the ground state noted by $(m,n)=(1,1)$, the points (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π) do not exist. Because the ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(1,1)},\eta_{ij}^{(1,1)})}\rangle$ has even number particles, it is also permitted by the projection. Such a state also carries a (0,0) crystal momentum.

Therefore there are two degenerate ground states on an even by odd lattice. Similarly topological degeneracy for $Z2$E state on an odd by even is also two. All those states carry a (0,0) crystal momentum. This corresponds to the second and third columns of Table II.

Last, let us discuss the topological degeneracy for $Z2$E state on an odd-by-odd lattice. For the ground state described by $(m,n)=(0,0)$, the $\boldsymbol{k}=(0,0)$ level is not occupied. The points $(\pi,0)$, $(0,\pi)$, and (π,π) do not exist. As a result, no particle occupies the high-symmetry points. The ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(0,0)},\eta_{ij}^{(0,0)})}\rangle$ has even number particles which is permitted by the projection.

For the ground state described by $(m,n)=(1,0)$, the $\boldsymbol{k}=(\pi,0)$ level is occupied by a ψ_1 particle. The points (0,0), $(0,\pi)$, and (π,π) do not exist. As a result, one particle occupies the high-symmetry points. Because the ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(1,0)},\eta_{ij}^{(1,0)})}\rangle$ has odd number particles, it is not permitted by the projection.

For the ground state described by $(m,n)=(0,1)$, the $\boldsymbol{k}=(0,\pi)$ level is occupied by a ψ_2 particle. The points (0,0), $(\pi,0)$, and (π,π) do not exist. As a result, one particle occupies the high-symmetry points. Because the ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(1,0)},\eta_{ij}^{(1,0)})}\rangle$ has odd number particles, it is not permitted by the projection.

For the ground state described by $(m,n)=(1,1)$, the $\boldsymbol{k}=(\pi,\pi)$ level is occupied by a ψ_1 and a ψ_2 particle. The points $(\pi,0)$, $(0,\pi)$, and (0,0) do not exist. As a result, two particles occupy the high-symmetry points. The ground state $|\Psi_{\text{mean}}^{(u_{ij}^{(1,1)},\eta_{ij}^{(1,1)})}\rangle$ has even number particles, so the state is permitted by the projection.

In conclusion, $Z2$E state has fourfold degeneracy on an even-by-even lattice and twofold degeneracy on an even-by-odd lattice, odd-by-even lattice, or odd-by-odd lattice. The crystal momenta of those ground states are given by Table II.

2. Quantization for the mutual $U(1)\times U(1)$ CS theory

To calculate the topological properties for the ground states of the mutual $U(1)\times U(1)$ CS theories, one needs to quantize the gauge fields. We will choose the temporal gauge $A_0=0$. In the temporal gauge, the physical degrees of freedom are described by (A_x,A_y) and (a_x,a_y). We will concentrate on the dynamics of $\theta_{x,y}$ and $\Theta_{x,y}$.

After the mode expansion, the effective Lagrangian

$$\mathcal{L}_{\text{eff}}=-\frac{1}{4e_a^2}(f_{\mu\nu})^2-\frac{1}{4e_A^2}(F_{\mu\nu})^2+\frac{1}{\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \quad \text{(A1)}$$

can be written as

$$L=\frac{1}{2}M_x\dot{\Theta}_x^2+\frac{1}{2}M_y\dot{\Theta}_y^2+\frac{1}{2}m_x\dot{\theta}_x^2+\frac{1}{2}m_y\dot{\theta}_y^2-\frac{1}{2\pi}\Theta_x\dot{\theta}_y -\frac{1}{2\pi}\Theta_y\dot{\theta}_x+\frac{1}{2\pi}\theta_y\dot{\Theta}_x+\frac{1}{2\pi}\theta_x\dot{\Theta}_y+\dots,$$

where $(A_{\mathbf{k}}^x,A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x,a_{\mathbf{k}}^y)$ represent the terms that contain only the $\mathbf{k}\neq 0$ modes. The masses are given as $M_x=\frac{1}{e_A^2}\frac{L_y}{L_x}$, $M_y=\frac{1}{e_A^2}\frac{L_x}{L_y}$ and $m_x=\frac{1}{e_a^2}\frac{L_y}{L_x}$, $m_y=\frac{1}{e_a^2}\frac{L_x}{L_y}$. Because the existence of the mass gap, the degree freedoms for gauge fields with nonzero momentum $(A_{\mathbf{k}}^x,A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x,a_{\mathbf{k}}^y)$ have nothing to do with the low-energy physics.

From the effective Lagrangian, one can define the conjugate momentum for (Θ_x,Θ_y) and (θ_x,θ_y),

$$P_{\Theta_x}=\frac{\partial L_{\text{eff}}}{\partial\dot{\Theta}_x}=M_x\dot{\Theta}_x+\frac{\theta_y}{2\pi},$$

$$P_{\Theta_y}=\frac{\partial L_{\text{eff}}}{\partial\dot{\Theta}_y}=M_y\dot{\Theta}_y-\frac{\theta_x}{2\pi},$$

$$p_{\theta_x}=\frac{\partial L_{\text{eff}}}{\partial\dot{\theta}_x}=m_x\dot{\theta}_x+\frac{\Theta_y}{2\pi},$$

$$p_{\theta_y}=\frac{\partial L_{\text{eff}}}{\partial\dot{\theta}_y}=m_y\dot{\theta}_y-\frac{\Theta_x}{2\pi}.$$

Using the conjugate momentum we write down the following effective Hamiltonian to describe the low-energy physics of the mutual $U(1)\times U(1)$ CS theory

$$H_{\text{eff}}=\frac{\left(P_{\Theta_x}-\frac{\theta_y}{2\pi}\right)^2}{2M_x}+\frac{\left(p_{\theta_y}+\frac{\Theta_x}{2\pi}\right)^2}{2m_x}+\frac{\left(P_{\Theta_y}+\frac{\theta_x}{2\pi}\right)^2}{2M_y}+\frac{\left(p_{\theta_x}-\frac{\Theta_y}{2\pi}\right)^2}{2m_x}.$$

By choosing different Landau gauges, the effective Hamiltonian can be rewritten as

$$H_{\text{eff}}=\frac{\left(P_{\Theta_x}-\frac{\theta_y}{\pi}\right)^2}{2M_x}+\frac{p_{\theta_y}^2}{2m_y}+\frac{\left(p_{\theta_x}-\frac{\Theta_y}{\pi}\right)^2}{2m_x}+\frac{P_{\Theta_y}^2}{2M_y}$$

or

$$H_{\text{eff}}=\frac{P_{\Theta_x}^2}{2M_x}+\frac{\left(p_{\theta_y}+\frac{\Theta_x}{\pi}\right)^2}{2m_y}+\frac{p_{\theta_x}^2}{2m_x}+\frac{\left(P_{\Theta_y}+\frac{\theta_x}{\pi}\right)^2}{2M_y}.$$

As a result, the low-energy properties of the $U(1)\times U(1)$ CS theory is described by the above Hamiltonian, which is a quantum mechanics model of two particles on a plane in magnetic field. Then one can obtain the topological degeneracy of the mutual $U(1)\times U(1)$ CS theory from the Landau degeneracy of the corresponding quantum mechanics model.

3. Topological degeneracy and crystal momenta of Z2E-type mutual $U(1)\times U(1)$ CS theory

In this section, we will calculate the topological degeneracy and crystal momenta of Z2E-type mutual $U(1)\times U(1)$ CS theory. Z2E-type mutual $U(1)\times U(1)$ CS theory is characterized by a special realization of the lattice translation symmetry defined in Eq. (14). We will see that such a realization of the translation symmetry leads to fourfold degeneracy on even-by-even lattice, and twofold degeneracy on even-by-odd, odd-by-even, and odd-by-odd lattices.

Let us first calculate the ground-state degeneracy of the Z2E state on an even-by-even lattice in detail.

We note that a_x and $a_x+\frac{2\pi}{L_x}$ are related by a $U(1)$ gauge transformation. Thus $\theta_x=0$ and $\theta_x=2\pi$ are also related by a $U(1)$ gauge transformation, which implies that $\theta_x=0$ and $\theta_x=2\pi$ should be viewed as the same point. Similarly each of the three pairs, $\theta_y=0$ and $\theta_y=2\pi$, $\Theta_x=0$ and $\Theta_x=2\pi$, $\Theta_y=0$ and $\Theta_y=2\pi$, also should be viewed as the same point. Thus the above Hamiltonian describes two particles, each moves on a $2\pi\times 2\pi$ torus. Each particle also see 4π flux through the torus.

The first particle is described by (Θ_x,θ_y). Since there are two units of flux through the torus, the ground states for the first particle has a degeneracy $D_{(\Theta_x,\theta_y)}=2$. Similarly, the ground states for the second particle also has a degeneracy $D_{(\Theta_y,\theta_x)}=2$.

As a result, for the Z2E-type mutual $U(1)\times U(1)$ CS theory, the ground states have fourfold degeneracy on an even-by-even lattice

$$D=D_{(\Theta_x,\theta_y)}D_{(\Theta_y,\theta_x)}=2\times 2=4. \tag{A2}$$

Moreover, the wave-functions Ψ for the four ground states with degenerate energy are given as $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$,

$$\Psi_1\simeq\exp\left[-\frac{1}{4\pi}\theta_y^2\right]\exp\left[-\frac{1}{4\pi}\Theta_y^2\right],$$

$$\Psi_2\simeq e^{-\mathrm{i}\Theta_x}\exp\left[-\frac{1}{4\pi}(\theta_y-\pi)^2\right]\exp\left[-\frac{1}{4\pi}\Theta_y^2\right],$$

$$\Psi_3\simeq e^{-\mathrm{i}\theta_x}\exp\left[-\frac{1}{4\pi}\theta_y^2\right]\exp\left[-\frac{1}{4\pi}(\Theta_y-\pi)^2\right],$$

$$\Psi_4\simeq e^{-\mathrm{i}\theta_x}e^{-\mathrm{i}\Theta_x}\exp\left[-\frac{1}{4\pi}(\theta_y-\pi)^2-\frac{1}{4\pi}(\Theta_y-\pi)^2\right]. \tag{A3}$$

Now let us calculate the crystal momentum for the fourfold-degenerate ground states. For the Z2E-type mutual $U(1)\times U(1)$ CS theory, the translation operations T_i are known as

$$T_i^{-1}A_jT_i=a_j,\quad T_i^{-1}a_jT_i=A_j.$$

Thus we have the translation operation for its zero modes (Θ_x,Θ_y) and (θ_x,θ_y):

$$T_x^{-1}\theta_xT_x=\Theta_x,$$

$$T_y^{-1}\theta_yT_y=\Theta_y,$$

$$T_x^{-1}\theta_yT_x=\Theta_y,$$

$$T_y^{-1}\theta_xT_y=\Theta_x.$$

Under the translation operators, we have

$$T_x|j\rangle=|j\rangle,$$

$$T_y|j\rangle = |j\rangle,$$

$$j = 1,4.$$

$$T_x|2\rangle = |3\rangle, \quad T_y|2\rangle = |3\rangle,$$

$$T_x|3\rangle = |2\rangle, \quad T_y|3\rangle = |2\rangle.$$

So $|2\rangle$ and $|3\rangle$ cannot be the eigenstates for the ground state. Instead, the eigenstates for the ground state are given as $|2'\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ and $|3'\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$. For $|2'\rangle$ and $|3'\rangle$, the eigenvalues of the translation operators are given as

$$T_x|2'\rangle = |2'\rangle, \quad T_y|2'\rangle = |2'\rangle,$$

$$T_x|3'\rangle = e^{i\pi}|3'\rangle, \quad T_y|3'\rangle = e^{i\pi}|3'\rangle.$$

As a result, on an even-by-even lattice, the crystal momentum of the E-type mutual $U(1)\times U(1)$ CS theory is $(K_x,K_y)=(0,0)$ for the ground states $|1\rangle$, $|2'\rangle$, $|4\rangle$, and $(K_x,K_y)=(\pi,\pi)$ for the ground state $|3'\rangle$.

For other cases, on an even-by-odd, odd-by-even, or odd-by-odd lattice, the situations are changed. Because for odd number rows along the x or y axis, one gauge field A_μ (a_μ) will turn into the other one a_μ (A_μ). For example, on a $L_x \times L_y$ even-by-odd lattice (L_x is an even number and L_y is an odd number), under such a twisted boundary condition for odd number L_y, one has

$$A_\mu(x,y+L_y) = a_\mu(x,y),$$

$$a_\mu(x,y+L_y) = A_\mu(x,y),$$

$$A_\mu(x+L_x,y) = A_\mu(x,y),$$

$$a_\mu(x+L_x,y) = a_\mu(x,y). \tag{A4}$$

The quantization for gauge fields in Eq. (15) cannot be applied to the gauge fields under a twisted boundary condition.

Now after putting the mutual $U(1)\times U(1)$ CS theory on a $L_x \times (2L_y)$ even-by-odd lattice, we have a periodic boundary condition,

$$A_\mu(x,y+2L_y) = A_\mu(x,y), a_\mu(x,y+2L_y) = a_\mu(x,y).$$

In the temporal gauge, $A_0=0$, and on such even-by-even lattice, we can expand the fluctuations for the gauge fields as

$$(A_x,A_y) = \left(\frac{1}{L_x}\Theta_x + \sum_{\mathbf{k}} A^x_{\mathbf{k}} e^{i\check{x}\cdot\mathbf{k}}, \frac{1}{2L_y}\Theta_y + \sum_{\mathbf{k}} A^y_{\mathbf{k}} e^{i\check{x}\cdot\mathbf{k}}\right), \tag{A5}$$

$$(a_x,a_y) = \left(\frac{1}{L_x}\theta_x + \sum_{\mathbf{k}} a^x_{\mathbf{k}} e^{i\check{x}\cdot\mathbf{k}}, \frac{1}{2L_y}\theta_y + \sum_{\mathbf{k}} a^y_{\mathbf{k}} e^{i\check{x}\cdot\mathbf{k}}\right), \tag{A6}$$

where $\mathbf{k}=(k_x,k_y)=(\frac{2\pi}{L_x}n_x, \frac{\pi}{L_y}n_y)$ where $n_{x,y}$ are integers. $(A^x_{\mathbf{k}}, A^y_{\mathbf{k}})$ and $(a^x_{\mathbf{k}}, a^y_{\mathbf{k}})$ are the gauge fields with nonzero momentum, and (Θ_x,Θ_y) and (θ_x,θ_y) are the zero modes with zero momentum for the gauge fields A_i and a_i. However, $A^i_{\mathbf{k}}$ and $a^i_{\mathbf{k}}$ (Θ_i and θ_i) are not independent and have constraints; to obey the original twisted boundary condition in Eq. (A4), we must have

$$A^i_{\mathbf{k}} = a^i_{\mathbf{k}} e^{iL_y\cdot k_y} = a^i_{\mathbf{k}} e^{i\cdot\pi n_y},$$

$$\Theta_i = \theta_i. \tag{A7}$$

To calculate the topological degeneracy, we map the original mutual $U(1)\times U(1)$ CS theory on even-by-odd lattice to two-particle quantum mechanics model on a torus in a magnetic field $\frac{1}{\pi}$. In the "Landau gauge," the effective Hamiltonian of the two-particle quantum mechanics model is given as

$$H_{\text{eff}} = \frac{\left(P_{\Theta_x} - \frac{\theta_y}{2\pi}\right)^2}{2M_x} + \frac{\left(p_{\theta_y} + \frac{\Theta_x}{2\pi}\right)^2}{2m_x} + \frac{\left(P_{\Theta_y} + \frac{\theta_x}{2\pi}\right)^2}{2M_y} + \frac{\left(p_{\theta_x} - \frac{\Theta_y}{2\pi}\right)^2}{2m_x},$$

where $M_x = \frac{1}{e_A^2}\frac{2L_y}{L_x}$, $M_y = \frac{1}{e_A^2}\frac{L_x}{2L_y}$ and $m_x = \frac{1}{e_a^2}\frac{2L_y}{L_x}$, $m_y = \frac{1}{e_a^2}\frac{L_x}{2L_y}$. However, because of the constraint in Eq. (A7), the two particles (θ_x,θ_y) and (Θ_x,Θ_y) are bound into a single particle! As a result, there are two degenerate ground states instead of four. In addition, we can write down the wave functions for the two ground states in the Landau gauge with topological degeneracy: for the wave-function $|1\rangle$,

$$\Psi_1 \simeq e^{-\frac{1}{4\pi}\theta_y^2} = e^{-\frac{1}{4\pi}\Theta_y^2},$$

and the wave-function $|2\rangle$,

$$\Psi_2 \simeq e^{-i\Theta_x} e^{-\frac{1}{4\pi}(\theta_y - \pi)^2} = e^{-i\theta_x} e^{-\frac{1}{4\pi}(\Theta_y - \pi)^2}.$$

Now let us calculate the crystal momentum for the twofold-degenerate ground states. The ground states are invariant under the translation operations

$$T_x|j\rangle = |j\rangle,$$

$$T_y|j\rangle = |j\rangle,$$

$$j = 1,2.$$

Then the crystal momentum (K_x,K_y) is (0,0) for the E-type mutual $U(1)\times U(1)$ CS theory on an even-by-odd lattice.

Furthermore, using the same method, we calculated the topological degeneracies and the crystal momenta for the ground states of the $Z2$E-type mutual $U(1)\times U(1)$ CS theory on an odd-by-even or odd-by-odd lattice. The results are similar to those on an even-by-odd lattice: the ground states have twofold degeneracy and $(K_x,K_y)=(0,0)$.

In summary, all the low-energy physical properties for the $Z2$E-type $U(1)\times U(1)$ Chern-Simons theory match that for the $Z2$E topological ordered state.

4. Topological degeneracy and crystal momenta of Z2A-type mutual $U(1)\times U(1)$ CS theory

In this section, we will calculate the topological degeneracy and crystal momenta for Z2A-type mutual $U(1)\times U(1)$ CS theory. Z2A-type mutual $U(1)\times U(1)$ CS theory are characterized by a special realization of the lattice translation symmetry defined in Eq. (13). On even-by-even, odd-by-even, or even-by-odd lattice, the low-energy properties of the $U(1)\times U(1)$ Chern-Simons theory is reduced into a quantum mechanics model of two particles on a plane in magnetic field. In addition, the translation symmetry defined in Eq. (13) leads to the nontrivial crystal momenta of Z2A state. However, on an odd-by-odd lattice, the situation is different. We will show that the four degenerate ground states are all forbidden by translation invariance. As a result, an emergent nonzero background charge leads to an infinity degeneracy on odd-by-odd lattice for the Z2A state. In the following, we will show the exotic properties of Z2A state in detail.

The effective Hamiltonian to describe the low-energy physics of the Z2A type the mutual $U(1)\times U(1)$ CS theory can be written in the Landau gauge as

$$H_{\mathrm{eff}}=\frac{\left(P_{\Theta_x}-\frac{\theta_y}{\pi}\right)^2}{2M_x}+\frac{p_{\theta_y}^2}{2m_y}+\frac{\left(p_{\theta_x}-\frac{\Theta_y}{\pi}\right)^2}{2m_x}+\frac{P_{\Theta_y}^2}{2M_y}.$$

It is noted that there exists the Heisenberg algebra for effective Hamiltonian. The "magnetic" translation operators $U_{\theta_x}=e^{\pi i(p_{\theta_x}+\Theta_y/\pi)}$ and $U_{\Theta_y}=e^{\pi i(p_{\Theta_y}+\theta_x/\pi)}$ consist of the Heisenberg algebra

$$U_{\theta_x}U_{\Theta_y}=e^{i\pi}U_{\Theta_y}U_{\theta_x}.$$

Because the Hamiltonian is invariant for the operations U_{θ_x} and U_{Θ_y},

$$U_{\theta_x}^{-1}HU_{\theta_x}=H,$$

$$U_{\Theta_y}^{-1}HU_{\Theta_y}=H,$$

the ground states are the eigenstates of U_{θ_x} and U_{Θ_y}. So one can draw a conclusion from the Heisenberg algebra that the ground states have two-degeneracy for (θ_x,Θ_y). On the other hand, for (Θ_x,θ_y), one can do the same calculation. So the ground states have two degeneracy for (θ_y,Θ_x) which is characterized by the eigenstates of $U_{\theta_y}=e^{\pi i(p_{\theta_y}+\Theta_x/\pi)}$ and $U_{\Theta_x}=e^{\pi i(p_{\Theta_x}+\theta_y/\pi)}$. As a result, for the Z2A-type mutual $U(1)\times U(1)$ CS theory, the ground states have fourfold degeneracy

$$D=D_{(\Theta_x,\theta_y)}D_{(\Theta_y,\theta_x)}=2\times 2=4. \tag{A8}$$

We denote the four ground states with topological degeneracy as $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$,

$$U_{\theta_x}|1\rangle=|1\rangle,$$

$$U_{\theta_x}|2\rangle=|2\rangle,$$

$$U_{\theta_x}|3\rangle=e^{i\pi}|3\rangle,$$

$$U_{\theta_x}|4\rangle=e^{i\pi}|4\rangle,$$

and

$$U_{\Theta_y}|1\rangle=|1\rangle,$$

$$U_{\Theta_y}|2\rangle=e^{i\pi}|2\rangle,$$

$$U_{\Theta_y}|3\rangle=|3\rangle,$$

$$U_{\Theta_y}|4\rangle=e^{i\pi}|4\rangle.$$

Now let us calculate the crystal momentum for the fourfold-degenerate ground states. For the Z2A-type mutual $U(1)\times U(1)$ Chern-Simons theory, the translation operations for the gauge fields are given by Eq. (13). The translation operations for zero modes of the gauge fields are given as Eq. (13)

$$T_x^{-1}\Theta_yT_x=\Theta_y,$$

$$T_y^{-1}\Theta_xT_y=\Theta_x,$$

$$T_x^{-1}\Theta_yT_x=\Theta_y+L_y\pi,$$

$$T_y^{-1}\Theta_xT_y=\Theta_x,$$

and

$$T_x^{-1}\theta_yT_x=\theta_y,$$

$$T_y^{-1}\theta_xT_y=\theta_x+L_x\pi,$$

$$T_x^{-1}\theta_xT_x=\theta_x,$$

$$T_y^{-1}\theta_yT_y=\theta_y.$$

As a result, the real ground states can be labeled by the eigenvalues of U_{θ_x} (or U_{θ_y}, U_{Θ_y}, U_{Θ_x}) which are 1 and -1. We denote the four ground states with topological degeneracy as $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$,

$$U_{\theta_x}|1\rangle=|1\rangle,$$

$$U_{\theta_x}|2\rangle=|2\rangle,$$

$$U_{\theta_x}|3\rangle=e^{i\pi}|3\rangle,$$

$$U_{\theta_x}|4\rangle=e^{i\pi}|4\rangle.$$

First, on an even-by-even lattice, the translation operations for its zero modes lead to trivial results

$$T_x^{-1}\Theta_yT_x=\Theta_y,$$

$$T_y^{-1}\Theta_xT_y=\Theta_x,$$

$$T_x^{-1}\Theta_y T_x = \Theta_y,$$

$$T_y^{-1}\Theta_x T_y = \Theta_x.$$

and

$$T_x^{-1}\theta_y T_x = \theta_y,$$

$$T_y^{-1}\theta_x T_y = \theta_x,$$

$$T_x^{-1}\theta_x T_x = \theta_x,$$

$$T_y^{-1}\theta_y T_y = \theta_y,$$

From them, we have

$$T_x|j\rangle = |j\rangle,$$

$$T_y|j\rangle = |j\rangle,$$

$$j = 1,2,3,4.$$

Then the crystal momentum (K_x, K_y) of the fourfold-degenerate ground states $|j\rangle$ is (0,0).

Second on an odd-by-even lattice (L_x is odd number and L_y is even number), the translation operations are given as

$$T_x^{-1}\theta_y T_x = \theta_y,$$

$$T_y^{-1}\theta_x T_y = \theta_x + \pi,$$

$$T_x^{-1}\theta_x T_x = \theta_x,$$

$$T_y^{-1}\theta_y T_y = \theta_y,$$

and

$$T_x^{-1}\Theta_y T_x = \Theta_y,$$

$$T_y^{-1}\Theta_x T_y = \Theta_x,$$

$$T_x^{-1}\Theta_x T_x = \Theta_x,$$

$$T_y^{-1}\Theta_y T_y = \Theta_y.$$

Now the translation operator T_y turns into the magnetic translation operator $U_{\theta_x} = e^{\pi i(p_{\theta_x} + \Theta_y/\pi)}$,

$$T_y|\mathrm{i}\rangle = U_{\theta_x}|\mathrm{i}\rangle = e^{\pi i(p_{\theta_x} + \Theta_y/\pi)}|\mathrm{i}\rangle, \quad i = 1,2,3,4.$$

Under the translation operations on the wave functions in Eq. (A3), we have

$$T_x|1\rangle = |1\rangle,$$

$$T_x|2\rangle = |2\rangle,$$

$$T_x|3\rangle = |3\rangle,$$

$$T_x|4\rangle = |4\rangle,$$

and

$$T_y|1\rangle = U_{\theta_x}|1\rangle = |1\rangle,$$

$$T_y|2\rangle = U_{\theta_x}|2\rangle = |2\rangle,$$

$$T_y|3\rangle = U_{\theta_x}|2\rangle = e^{\mathrm{i}\pi}|3\rangle,$$

$$T_y|4\rangle = U_{\theta_x}|2\rangle = e^{\mathrm{i}\pi}|4\rangle.$$

Using the same method, we can obtain that the crystal momentum of the two ground states $|1\rangle$ and $|2\rangle$ is (0,0). The crystal momentum of the other two ground states $|3\rangle$ and $|4\rangle$ is $(0,\pi)$.

Third, on an even-by-odd lattice (L_x is even number and L_y is odd number), the translation operations for its zero modes lead to nontrivial results

$$T_x^{-1}\Theta_y T_x = \Theta_y,$$

$$T_y^{-1}\Theta_x T_y = \Theta_x,$$

$$T_x^{-1}\Theta_y T_x = \Theta_y + \pi,$$

$$T_y^{-1}\Theta_x T_y = \Theta_x,$$

and

$$T_x^{-1}\theta_y T_x = \theta_y,$$

$$T_y^{-1}\theta_x T_y = \theta_x,$$

$$T_x^{-1}\theta_x T_x = \theta_x,$$

$$T_y^{-1}\theta_y T_y = \theta_y.$$

Then the translation operator T_x turns into the magnetic translation operator $U_{\Theta_y} = e^{\pi i(p_{\Theta_y} + \theta_x/\pi)}$,

$$T_x|\mathrm{i}\rangle = U_{\Theta_y}|\mathrm{i}\rangle = e^{\pi i(p_{\Theta_y} + \theta_x/\pi)}|\mathrm{i}\rangle, \quad i = 1,2,3,4.$$

From them, we have

$$T_x|1\rangle = U_{\Theta_y}|1\rangle = |1\rangle,$$

$$T_x|2\rangle = U_{\Theta_y}|2\rangle = e^{\mathrm{i}\pi}|2\rangle,$$

$$T_x|3\rangle = U_{\Theta_y}|3\rangle = |3\rangle,$$

$$T_x|4\rangle = U_{\Theta_y}|4\rangle = e^{\mathrm{i}\pi}|4\rangle,$$

and

$$T_y|1\rangle = |1\rangle,$$

$$T_y|3\rangle = |3\rangle,$$

$$T_y|2\rangle = |2\rangle,$$

$$T_y|4\rangle = |4\rangle.$$

The crystal momentum of two ground states $|1\rangle$ and $|3\rangle$ is (0,0). The crystal momentum of the other two ground states $|4\rangle$ and $|2\rangle$ is $(\pi,0)$.

Fourth, for L_x and L_y are all odd numbers (on an odd-by-odd lattice), the translation operations become

$$T_x^{-1}\Theta_y T_x = \Theta_y + \pi,$$

$$T_y^{-1}\Theta_x T_y = \Theta_x,$$

$$T_x^{-1}\Theta_x T_x = \Theta_x,$$

$$T_y^{-1}\Theta_y T_y = \Theta_y,$$

and

$$T_x^{-1}\theta_y T_x = \theta_y,$$

$$T_y^{-1}\theta_x T_y = \theta_x + \pi,$$

$$T_x^{-1}\theta_x T_x = \theta_x,$$

$$T_y^{-1}\theta_y T_y = \theta_y.$$

Then the translation operators T_x and T_y turn into the magnetic translation operator $U_{\Theta_y} = e^{\pi i(p_{\Theta_y}+\theta_x/\pi)}$ and $U_{\theta_x} = e^{\pi i(p_{\theta_x}+\Theta_y/\pi)}$,

$$T_x|\mathrm{i}\rangle = U_{\Theta_y}|\mathrm{i}\rangle = e^{\pi i(p_{\Theta_y}+\theta_x/\pi)}|\mathrm{i}\rangle,$$

$$T_y|\mathrm{i}\rangle = U_{\theta_x}|\mathrm{i}\rangle = e^{\pi i(p_{\theta_x}+\Theta_y/\pi)}|\mathrm{i}\rangle, \quad i = 1,2,3,4.$$

Now T_x and T_y must obey the Heisenberg algebra for U_{Θ_y} and U_{θ_x}

$$T_x T_y = e^{\mathrm{i}\pi} T_y T_x. \tag{A9}$$

On the other hand, the translation symmetry of the system leads to the commutation relationship between T_x and T_y

$$T_x T_y = T_y T_x. \tag{A10}$$

The only solution to Eqs. (A9) and (A10) is $|\mathrm{i}\rangle \equiv 0$. That is, there do not exist the four degenerate ground states at all. We can see that for the real ground states, the A_μ and a_μ charges for the excitations cannot be zero on an odd by odd lattice. So the nonzero background charge leads to an infinity degeneracy on odd-by-odd lattice for the $Z2$A type mutual $U(1)\times U(1)$ CS theory.

As a result, all the low-energy physical properties for the $Z2$A-type $U(1)\times U(1)$ Chern-Simons theory match that for the $Z2$A topological ordered state.

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